

Forward stochastic reachability analysis for uncontrolled linear systems using Fourier Transforms

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ABSTRACT

We propose a scalable method for forward stochastic reachability analysis for uncontrolled linear systems with affine disturbance. Our method uses Fourier transforms to efficiently compute the forward stochastic reach probability measure (density) and the forward stochastic reach set. This method is applicable to systems with bounded or unbounded disturbance sets. We also examine the properties of the forward stochastic reach set and its probability density. Motivated by the problem of a robot attempting to capture a stochastically moving, non-adversarial target, we demonstrate our method on two simple examples. Where traditional approaches provide approximations, our method provides exact analytical expressions for the densities.

CCS Concepts

•Theory of computation → Stochastic control and optimization; *Convex optimization*; •Computing methodologies → *Control methods*; Computational control theory;

Keywords

Stochastic reachability; Fourier transform; Convex optimization

1. INTRODUCTION

Reachability analysis of discrete-time dynamical systems with stochastic disturbance input is an established tool to provide probabilistic assurances of safety or performance and has been applied in several domains, including anesthesia delivery [1], motion planning in robotics [2], spacecraft docking [3], and planetary missions [4]. The computation of stochastic reachable and viable sets has been formulated within a dynamic programming framework [5,6] that generalizes to discrete-time stochastic hybrid systems, and suffers from the well-known curse of dimensionality [7]. Recent work in computing stochastic reachable and viable sets aims

to circumvent these computational challenges, through approximate dynamic programming [8–10], Gaussian mixtures [9], particle filters [3,10] and convex chance-constrained optimization [3,11]. These methods have been applied to systems that are at most 6-dimensional [8] – far beyond the scope of what is possible with dynamic programming, but are not scalable to larger and more realistic scenarios.

We focus in particular on the forward stochastic reachable set, which is those set of states which the state can reach at some future time from a known initial condition, with a known, desired likelihood, and not on the stochastic viable set or the stochastic reach-avoid set. For LTI systems with bounded disturbances, the forward stochastic reachable set can be computed with established methods from the verification literature [12–15]. However, these methods return a trivial result with unbounded disturbances and do not address the forward stochastic reach probability measure, which provides the likelihood of reaching a given state.

We present a scalable method to perform forward stochastic reachability analysis of linear time-invariant systems with stochastic dynamics, that is, a method to compute the forward stochastic reachable set as well as its probability measure. We show that Fourier transforms can be used to provide exact reachability analysis, for systems with bounded or unbounded disturbances. We provide both iterative and analytical expressions for the probability density, and show that explicit expressions can be derived in some cases.

We are motivated by a particular application: pursuit of a dynamic, non-adversarial target [16]. Such a scenario may arise in e.g., rescue of a lost first responder in a building on fire [17], capture of a non-aggressive UAV in an urban environment [18], or other non-antagonistic situations. Solutions for an adversarial target, based in a two-person, zero-sum differential game, can accommodate bounded disturbances with unknown stochasticity [19–23], but will be conservative for a non-adversarial target. Related work provides a solution for multiple pursuers capturing an adversarial evader in [24–26]. We seek real-time compatible solutions that synthesize an optimal controller for the non-adversarial scenario, by exploiting the forward reachable set and probability measure for the target. We analyze the convexity properties of the forward stochastic reach probability density and sets, and propose a convex optimization problem to provide the exact probabilistic guarantee of success and the corresponding optimal controller.

The main contributions of this paper are: 1) a method to efficiently compute the forward stochastic reach sets and the corresponding probability measure for linear systems with

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uncertainty using Fourier transforms, 2) convexity properties of the forward stochastic reach probability measure and sets, and 3) a convex formulation to maximize the probability of capture of a non-adversarial target with stochastic dynamics, that exploits forward stochastic reachability analysis.

The paper is organized as follows: We define the forward stochastic reachability problem and review some properties from probability theory and Fourier analysis in Section 2. Section 3 formulates the forward stochastic reachability analysis for linear systems using Fourier transforms, and provides convexity results for the probability measure and the stochastic reachable set. We apply the proposed method to solve the controller synthesis problem in Section 4, and provide conclusions and directions for future work in Section 5.

2. PRELIMINARIES AND PROBLEM FORMULATION

In this section, we review some properties from probability theory and Fourier analysis relevant for our discussion and setup the problems. For detailed discussions on probability theory, see [27–30], and on Fourier analysis, see [31]. We denote random vectors with bold case and non-random vectors with an overline.

2.1 Probability Theory Preliminaries

A random vector $\mathbf{w} \in \mathbb{R}^p$ is defined in a probability space $(\mathcal{W}, \sigma(\mathcal{W}), \mathbb{P}_{\mathbf{w}})$. Given a sample space \mathcal{W} , the sigma-algebra $\sigma(\mathcal{W})$ provides a collection of measurable sets defined over \mathcal{W} . The sample space can be either countable (discrete random vector \mathbf{w}) or uncountable (continuous random vector \mathbf{w}). In this paper, we focus only on absolutely continuous random variables. For an absolutely continuous random vector, the probability measure defines a probability density function $\psi_{\mathbf{w}} : \mathbb{R}^p \rightarrow \mathbb{R}$ such that given a (Borel) set $\mathcal{S} \in \sigma(\mathcal{W})$, we have $\mathbb{P}_{\mathbf{w}}\{\mathbf{w} \in \mathcal{S}\} = \int_{\mathcal{S}} \psi_{\mathbf{w}}(\bar{z}) d\bar{z}$. Here, $d\bar{z}$ is short for $dz_1 dz_2 \dots dz_p$. Recall that linear transformations preserve measurability of spaces.

The characteristic function of a random vector $\mathbf{w} \in \mathbb{R}^p$ with probability density function $\psi_{\mathbf{w}}(\bar{z})$ is

$$\begin{aligned} \Psi_{\mathbf{w}}(\bar{\alpha}) &\triangleq \mathbb{E}_{\mathbf{w}}[\exp(j\bar{\alpha}^\top \mathbf{w})] \\ &= \int_{\mathbb{R}^p} e^{j\bar{\alpha}^\top \bar{z}} \psi_{\mathbf{w}}(\bar{z}) d\bar{z} = \mathcal{F}\{\psi_{\mathbf{w}}(\cdot)\}(-\bar{\alpha}) \end{aligned} \quad (1)$$

where $\mathcal{F}\{\cdot\}$ denotes the Fourier transformation operator and $\bar{\alpha} \in \mathbb{R}^p$. Given a characteristic function $\Psi_{\mathbf{w}}(\bar{\alpha})$, the density function can be computed as

$$\begin{aligned} \psi_{\mathbf{w}}(\bar{z}) &= \mathcal{F}^{-1}\{\Psi_{\mathbf{w}}(\cdot)\}(-\bar{z}) \\ &= \left(\frac{1}{2\pi}\right)^p \int_{\mathbb{R}^p} e^{-j\bar{\alpha}^\top \bar{z}} \Psi_{\mathbf{w}}(\bar{\alpha}) d\bar{\alpha} \end{aligned} \quad (2)$$

where $\mathcal{F}^{-1}\{\cdot\}$ denotes the inverse Fourier transformation operator and $d\bar{\alpha}$ is short for $d\alpha_1 d\alpha_2 \dots d\alpha_p$.

We define the $L^d(\mathbb{R}^p)$ spaces, $1 \leq d < \infty$, of measurable complex-valued functions with finite L^d norm. The L^d norm of a density $\psi_{\mathbf{w}}$ is $\|\psi_{\mathbf{w}}\|_d \triangleq \left(\int_{\mathbb{R}^p} |\psi_{\mathbf{w}}(\bar{z})|^d d\bar{z}\right)^{1/d}$ where $|\cdot|$ denotes the absolute value. Here, $L^1(\mathbb{R}^p)$ is the space of absolutely integrable functions, and $L^2(\mathbb{R}^p)$ is the

space of square-integrable functions. The Fourier transformation is defined for all functions in $L^1(\mathbb{R}^p)$ and all functions in $L^2(\mathbb{R}^p)$. Since probability densities are, by definition, in $L^1(\mathbb{R}^p)$, characteristic functions exist for every probability density [31, Section 1]. Let $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^p$ be random vectors with densities $\psi_{\mathbf{w}_1}$ and $\psi_{\mathbf{w}_2}$ and characteristic functions $\Psi_{\mathbf{w}_1}$ and $\Psi_{\mathbf{w}_2}$ respectively. By definition, $\psi_{\mathbf{w}_1}, \psi_{\mathbf{w}_2} \in L^1(\mathbb{R}^p)$. Let $\bar{z}, \bar{z}_1, \bar{z}_2, \bar{\alpha}, \bar{\alpha}_1, \bar{\alpha}_2 \in \mathbb{R}^p, \bar{\beta} \in \mathbb{R}^n$.

- P1) If $\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2$, then $\psi_{\mathbf{x}}(\bar{z}) = (\psi_{\mathbf{w}_1}(\cdot) * \psi_{\mathbf{w}_2}(\cdot))(\bar{z})$ and $\Psi_{\mathbf{x}}(\bar{\alpha}) = \Psi_{\mathbf{w}_1}(\bar{\alpha}) \Psi_{\mathbf{w}_2}(\bar{\alpha})$. Here, $*$ is the convolution operation. [29, Section 21.11]
- P2) If $\mathbf{x} = F\mathbf{w}_1 + G$ where $F \in \mathbb{R}^{p \times n}, G \in \mathbb{R}^n$ are matrices, $\Psi_{\mathbf{x}}(\bar{\beta}) = \exp(j\bar{\beta}^\top G) \Psi_{\mathbf{w}_1}(F^\top \bar{\beta})$ (from [29, Section 22.6] and [31, Equation 1.5]).
- P3) If \mathbf{w}_1 and \mathbf{w}_2 are independent vectors, then $\mathbf{x} = [\mathbf{w}_1^\top \mathbf{w}_2^\top]^\top$ has probability density $\psi_{\mathbf{x}}(\bar{y}) = \psi_{\mathbf{w}_1}(\bar{z}_1) \psi_{\mathbf{w}_2}(\bar{z}_2), \bar{y} = [\bar{z}_1^\top \bar{z}_2^\top]^\top \in \mathbb{R}^{2p}$ and characteristic function $\Psi_{\mathbf{x}}(\bar{\gamma}) = \Psi_{\mathbf{w}_1}(\bar{\alpha}_1) \Psi_{\mathbf{w}_2}(\bar{\alpha}_2), \bar{\gamma} = [\bar{\alpha}_1^\top \bar{\alpha}_2^\top]^\top \in \mathbb{R}^{2p}$ [29, Section 22.4].
- P4) The marginal probability density of any group of k components selected from the random vector \mathbf{w}_1 is obtained by setting the remaining $p - k$ Fourier variables in the characteristic function to zero [29, Section 22.4].

An additional assumption of square-integrability of the probability density of the random variable $\mathbf{w}_3 \in \mathbb{R}^p$ results in $\psi_{\mathbf{w}_3} \in L^1(\mathbb{R}^p) \cap L^2(\mathbb{R}^p)$. Along with Properties P1-P4, $\psi_{\mathbf{w}_3}$ satisfies the following property:

- P5) The Fourier transform preserves the inner product in $L^2(\mathbb{R}^p)$ [31]. Given a square-integrable function $h(\bar{z})$ with Fourier transform $H(\bar{\alpha}) = \mathcal{F}\{h(\cdot)\}(\bar{\alpha})$ and a square-integrable probability density $\psi_{\mathbf{w}_3}$,

$$\int_{\mathbb{R}^p} \psi_{\mathbf{w}_3}(\bar{z})^\dagger h(\bar{z}) d\bar{z} = \left(\frac{1}{2\pi}\right)^p \int_{\mathbb{R}^p} (\mathcal{F}\{\psi_{\mathbf{w}_3}(\cdot)\}(\bar{\alpha}))^\dagger H(\bar{\alpha}) d\bar{\alpha}$$

Here, \dagger denotes complex conjugation.

Lemma 1. For square-integrable $\psi_{\mathbf{w}_3}$ and h ,

$$\int_{\mathbb{R}^p} \psi_{\mathbf{w}_3}(\bar{z}) h(\bar{z}) d\bar{z} = \left(\frac{1}{2\pi}\right)^p \int_{\mathbb{R}^p} \Psi_{\mathbf{w}_3}(\bar{\alpha}) H(\bar{\alpha}) d\bar{\alpha}. \quad (3)$$

Proof: Follows from Property P5 and (2). Since probability densities are real functions, $(\psi_{\mathbf{w}_3}(\bar{z}))^\dagger = \psi_{\mathbf{w}_3}(\bar{z})$ and $(\mathcal{F}\{\psi_{\mathbf{w}_3}(\cdot)\}(\bar{\alpha}))^\dagger = (\Psi_{\mathbf{w}_3}(-\bar{\alpha}))^\dagger = \Psi_{\mathbf{w}_3}(\bar{\alpha})$ [29, Section 10.6]. ■

Given a Borel set $S \subseteq \mathbb{R}^p$, let h be the indicator function $h(\bar{z}) = 1_S(\bar{z})$ with $h(\bar{z}) = 1$ if $\bar{z} \in S$ and zero otherwise. Clearly, h is square-integrable, and Lemma 1 can be used to compute the probability of the random variable with a square-integrable probability density \mathbf{w}_3 taking a value in S .

2.2 Problem formulation

Consider the discrete-time linear time-invariant system,

$$\mathbf{x}[t+1] = A\mathbf{x}[t] + B\mathbf{w}[t] \quad (4)$$

with state $\mathbf{x}[t] \in \mathcal{X} \subseteq \mathbb{R}^n$, disturbance $\mathbf{w}[t] \in \mathcal{W} \subseteq \mathbb{R}^p$, and matrices A, B of appropriate dimensions. Let $\bar{x}_0 \in \mathcal{X}$

be the given initial state and T be the time horizon. The disturbance set \mathcal{W} is an uncountable set which can be either bounded or unbounded, and the random vector $\mathbf{w}[t]$ is defined in a probability space $(\mathcal{W}, \sigma(\mathcal{W}), \mathbb{P}_{\mathbf{w}})$. The random vector $\mathbf{w}[t]$ is assumed to be absolutely continuous with a known density function $\psi_{\mathbf{w}}$. The disturbance process $\mathbf{w}[\cdot]$ is assumed to be a random process with an independent and identical distribution (IID).

The dynamics in (4) are quite general and includes affine noise perturbed LTI discrete-time systems with known state-feedback based inputs or open-loop controllers. For time $\tau \in [1, T]$,

$$\mathbf{x}[\tau] = A^\tau \bar{x}_0 + \mathcal{C}_{n \times (\tau p)} \mathbf{W} \quad (5)$$

with $\mathcal{C}_{n \times (\tau p)} = [B \ AB \ A^2B \ \dots \ A^{\tau-1}B] \in \mathbb{R}^{n \times (\tau p)}$ and $\mathbf{W} = [\mathbf{w}^\top[0] \ \mathbf{w}^\top[1] \ \dots \ \mathbf{w}^\top[\tau-1]]^\top$ as a random vector defined by the sequence of random vectors $\{\mathbf{w}[t]\}_{t=0}^{\tau-1}$. For any given τ , the random vector \mathbf{W} is defined in the product space $(\mathcal{W}^\tau, \sigma(\mathcal{W}^\tau), \mathbb{P}_{\mathbf{W}})$ where $\mathcal{W}^\tau = \times_{t=0}^{\tau-1} \mathcal{W}$ and $\mathbb{P}_{\mathbf{W}} = \prod_{t=0}^{\tau-1} \mathbb{P}_{\mathbf{w}}$, $\psi_{\mathbf{W}} = \prod_{t=0}^{\tau-1} \psi_{\mathbf{w}}$ by the IID assumption of the random process $\mathbf{w}[\cdot]$. Let $\bar{Z} = [\bar{z}^\top[0] \ \bar{z}^\top[1] \ \dots \ \bar{z}^\top[\tau-1]]^\top \in \mathbb{R}^{\tau p}$ denote a realization of the random vector \mathbf{W} .

From (5), the state $\mathbf{x}[\cdot]$ is a random process with the random vector at each instant $\mathbf{x}[t]$ defined in the probability space $(\mathcal{X}, \sigma(\mathcal{X}), \mathbb{P}_{\mathbf{x}}^t)$ where the probability measure $\mathbb{P}_{\mathbf{x}}^t$ is induced from $\mathbb{P}_{\mathbf{W}}$. We denote the random process originating from \bar{x}_0 as $\boldsymbol{\xi}(\cdot; \bar{x}_0)$ where for all t , $\boldsymbol{\xi}(t; \bar{x}_0) = \mathbf{x}[t]$.

An iterative method for computing the forward stochastic reachable set and probability measure for (4) was proposed in [32]. However, for systems perturbed by continuous random variables, the numerical implementation of the iterative approach becomes erroneous for larger time instants τ due to the iterative numerical evaluation of improper integrals. Hence an alternative approach is needed to be numerically implementable.

Problem 1. *Given the dynamics (4) with initial state \bar{x}_0 , construct analytical expressions at time instant τ for*

1. *the set of all states reachable with non-zero probability (i.e., the forward stochastic reach set), and*
2. *the probability measure over the forward stochastic reach set (i.e., the forward stochastic reach probability measure)*

that do not require an iterative approach.

We are additionally interested in applying the scalable forward reachable set and probability measure to the problem of capturing a non-adversarial target. Specifically, we seek a convex formulation to the problem of capturing a non-adversarial target. This requires convexity of the forward stochastic reach set (FSR set) and concavity of the objective function defined on the probability of successful capture.

Problem 2. *For a finite time horizon, find a) a convex formulation for the maximization of the probability of capture of a non-adversarial target with known stochastic dynamics and initial state, and b) the resulting optimal controller that a deterministic robot must employ when there is a non-zero probability of capture.*

Problem 2.a. *Characterize the sufficient conditions for log-concavity of the FSRPM and convexity of the FSR set.*

3. FORWARD STOCHASTIC REACHABILITY ANALYSIS

For any $\tau \in [1, T]$ and $\bar{x}_\tau \in \mathcal{X}$, let $\mathcal{D}(\tau, \bar{x}_\tau, \bar{x}_0) \in \sigma(\mathcal{W}^\tau)$ be the event where the random vector \mathbf{W} drives the system to evolve from \bar{x}_0 to \bar{x}_τ by time τ , and $\mathbb{P}_{\mathbf{W}}\{\mathbf{W} \in \mathcal{D}(\tau, \bar{x}_\tau, \bar{x}_0)\} = \int_{\mathcal{D}(\tau, \bar{x}_\tau, \bar{x}_0)} \psi_{\mathbf{W}}(\bar{Z}) d\bar{Z}$ where $d\bar{Z}$ is short for $d\bar{z}_0 d\bar{z}_1 \dots d\bar{z}_\tau$. From (5), we have

$$\mathcal{D}(\tau, \bar{x}_\tau, \bar{x}_0) = \{\bar{Z} \in \mathcal{W}^\tau | \bar{x}_\tau = A^\tau \bar{x}_0 + \mathcal{C}_{n \times (\tau p)} \bar{Z}\}. \quad (6)$$

We define the forward stochastic reach set (FSR set) for (4) with initial condition \bar{x}_0 as

$$\text{FSR}_{\text{Reach}}(\tau, \bar{x}_0) = \{\bar{x}_\tau \in \mathcal{X} | \exists \mathcal{D}(\tau, \bar{x}_\tau, \bar{x}_0) \in \sigma(\mathcal{W}^\tau) \text{ s.t. } \mathbb{P}_{\mathbf{W}}\{\mathbf{W} \in \mathcal{D}(\tau, \bar{x}_\tau, \bar{x}_0)\} > 0\}. \quad (7)$$

From (5), the probability of reaching a set $S \in \sigma(\mathcal{X})$ at time τ when the initial state is \bar{x}_0 is defined using the forward stochastic reach probability measure $\mathbb{P}_{\mathbf{x}}^{\tau, \bar{x}_0}$,

$$\mathbb{P}_{\mathbf{x}}^{\tau, \bar{x}_0}\{\mathbf{x}[\tau] \in S\} = \mathbb{P}_{\mathbf{W}}\left\{\bigcup_{\bar{x}_\tau \in S} \{\mathbf{W} \in \mathcal{D}(\tau, \bar{x}_\tau, \bar{x}_0)\}\right\}. \quad (8)$$

Since linear transformations preserve measurability, a probability density function $\psi_{\mathbf{x}}(\cdot; t, \bar{x}_0)$ represents the FSRPM as

$$\mathbb{P}_{\mathbf{x}}^{\tau, \bar{x}_0}\{\mathbf{x}[t] \in S\} = \int_S \psi_{\mathbf{x}}(\bar{y}; t, \bar{x}_0) d\bar{y}, \quad \bar{y} \in \mathbb{R}^n. \quad (9)$$

We focus on the computation of the forward stochastic reach probability density (FSRPD), and use (9) to compute the probability measure.

We characterize the FSR set with following lemmas.

Lemma 2. $\text{FSR}_{\text{Reach}}(t, \bar{x}_0) = \{A^t \bar{x}_0\} \oplus \mathcal{C}_{n \times (tp)} \mathcal{W}^t$.

Proof: Follows from (5). ■

Lemma 3. $\text{FSR}_{\text{Reach}}(t, \bar{x}_0) = \{\bar{y} \in \mathcal{X} : \psi_{\mathbf{x}}(\bar{y}; t, \bar{x}_0) > 0\}$.

Proof: Follows from (7) and (8). ■

Lemma 2 extends the definition of the reach sets for uncontrolled linear systems [12–14] to the stochastic framework, and Lemma 3 defines the FSR set as the support of the corresponding probability density. Note that the equalities in Lemmas 2 and 3 would be *almost sure* equalities if the additional restriction of non-zero probabilities were not imposed in (7).

3.1 Iterative method for reachability analysis

In this subsection, we extend the iterative approach for forward stochastic reachability analysis proposed in [32] for a nonlinear discrete-time systems perturbed by discrete random variables to a linear discrete-time system perturbed by continuous random variables. This discussion helps to develop some of the proofs discussed later in the paper.

Assume that the system matrix A of (4) is invertible. This assumption holds for continuous-time systems which have been discretized via Euler method. From (4) and Property P1, we have

$$\psi_{\mathbf{x}}(\bar{y}; t+1, \bar{x}_0) = (\psi_{A\mathbf{x}}(\cdot; t, \bar{x}_0) * \psi_{B\mathbf{w}}(\cdot))(\bar{y}) \quad (10)$$

with $\psi_{A\mathbf{x}}(\bar{y}; t, \bar{x}_0) = (\det A)^{-1} \psi_{\mathbf{x}}(A^{-1}\bar{y}; t, \bar{x}_0)$ for $t \geq 1$, $\psi_{A\mathbf{x}}(\bar{y}; 0, \bar{x}_0) = \delta(\bar{y} - A\bar{x}_0)$ where $\delta(\cdot)$ is the Dirac-delta function, and $\psi_{B\mathbf{w}}$ as the probability density of the random vector $B\mathbf{w}$ [28]. Equation (10) and Lemma 3 provide

an iterative method to compute the FSRPD (FSRPM via (9)) and the FSR set respectively.

Note that (10) is an improper integral which must be solved iteratively. For densities whose convolution integrals are difficult to obtain analytically, we would need to rely on numerical integration techniques. Numerical evaluation of multi-dimensional improper integrals is computationally expensive [33]. Moreover, the numerical evaluation of this method will become increasingly erroneous for larger values of $t \in [1, T]$ due to the iterative definition. These disadvantages motivate the need to solve Problem 1 — an approach that provides analytical expressions of the FSRPD, and thereby reduces the number of numerical integrations to be performed. The iterative method performs well with discrete random variables as in [32] because discretization for computation can be exact, however, this is clearly not true when the disturbance set is uncountable.

3.2 Efficient reachability analysis via characteristic functions

We employ Fourier transformation to provide analytical expressions of the FSRPD at any instant $\tau \in [1, T]$. This method involves computing a single integral for the time instant of interest as opposed to the iterative approach in Subsection 3.1. We also show that for certain disturbance distributions like the Gaussian distribution, an explicit expression for the FSRPD can be obtained.

By Property P3 and the IID assumption on the random process $\mathbf{w}[\cdot]$, the characteristic function of the random vector \mathbf{W} is

$$\Psi_{\mathbf{W}}(\bar{\alpha}) = \prod_{t=0}^{t=\tau-1} \Psi_{\mathbf{w}}(\bar{\alpha}_t) \quad (11)$$

where $\bar{\alpha} = [\bar{\alpha}_0^\top \bar{\alpha}_1^\top \dots \bar{\alpha}_{\tau-1}^\top]^\top \in \mathbb{R}^{(\tau p)}$, $\bar{\alpha}_t \in \mathbb{R}^p$ for all $t \in [0, \tau-1]$. As seen in (5), the random vector \mathbf{W} concatenates the disturbance random process $\mathbf{w}[t]$ over $t \in [0, \tau-1]$.

Theorem 1. *For any time instant $\tau \in [1, T]$ and an initial state $\bar{x}_0 \in \mathcal{X}$, the forward stochastic reach probability density $\psi_{\mathbf{x}}(\cdot; \tau, \bar{x}_0)$ is given by*

$$\Psi_{\mathbf{x}}(\bar{\alpha}; \tau, \bar{x}_0) = \exp\left(j\bar{\alpha}^\top (A^\tau \bar{x}_0)\right) \Psi_{\mathbf{W}}(\mathcal{C}_{n \times (\tau p)}^\top \bar{\alpha}) \quad (12)$$

$$\psi_{\mathbf{x}}(\bar{y}; \tau, \bar{x}_0) = \mathcal{F}^{-1}\{\Psi_{\mathbf{x}}(\bar{\alpha}; \tau, \bar{x}_0)\}(-\bar{y}) \quad (13)$$

where $\bar{y} \in \mathcal{X}$, $\bar{\alpha} \in \mathbb{R}^{n \times 1}$.

Proof: Follows from Property P2, (5), and (2). ■

Theorem 1 and Lemma 3 provide an analytical expression for the FSRPD and the FSR set respectively, and thereby solve Problem 1. Note that the computation of the FSRPD via Theorem 1 does not require gridding of the state space, hence mitigating the curse of dimensionality. Using Theorem 1, we derive an explicit expression for the FSRPD of (4) when perturbed by a Gaussian random vector.

Proposition 1. *For $\tau \in [1, T]$, the FSRPD of a system (4) with initial condition \bar{x}_0 and noise process $\mathbf{w} \sim \mathcal{N}(\bar{\mu}_{\mathbf{w}}, \Sigma_{\mathbf{w}}) \in \mathbb{R}^p$ is*

$$\xi(\tau; \bar{x}_0) \sim \mathcal{N}(\bar{\mu}[\tau], \Sigma[\tau]) \quad (14)$$

where

$$\bar{\mu}[\tau] = A^\tau \bar{x}_0 + \mathcal{C}_{n \times (\tau p)}(\bar{\mathbf{I}}_{\tau \times 1} \otimes \bar{\mu}_{\mathbf{w}}), \quad (15)$$

$$\Sigma[\tau] = \mathcal{C}_{n \times (\tau p)}(I_\tau \otimes \Sigma_{\mathbf{w}})\mathcal{C}_{n \times (\tau p)}^\top. \quad (16)$$

Proof: For $\bar{\alpha} \in \mathbb{R}^p$, the characteristic function of a multivariate Gaussian random vector \mathbf{w} is given as [28, Section 9.3]

$$\Psi_{\mathbf{w}}(\bar{\alpha}) = \exp\left(j\bar{\alpha}^\top \bar{\mu}_{\mathbf{w}} - \frac{\bar{\alpha}^\top \Sigma_{\mathbf{w}} \bar{\alpha}}{2}\right). \quad (17)$$

From the IID assumption of $\mathbf{w}[\cdot]$, Property P3, and (17), the characteristic function of \mathbf{W} is

$$\begin{aligned} \Psi_{\mathbf{W}}(\bar{\alpha}) &= \prod_{t=0}^{t=\tau-1} \exp\left(j\bar{\alpha}_t^\top \bar{\mu}_{\mathbf{w}} - \frac{\bar{\alpha}_t^\top \Sigma_{\mathbf{w}} \bar{\alpha}_t}{2}\right) \\ &= \exp\left(j\bar{\alpha}^\top (\bar{\mathbf{I}}_{\tau \times 1} \otimes \bar{\mu}_{\mathbf{w}}) - \frac{\bar{\alpha}^\top (I_\tau \otimes \Sigma_{\mathbf{w}}) \bar{\alpha}}{2}\right) \end{aligned}$$

where $\bar{\alpha} = (\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{\tau-1}) \in \mathbb{R}^{(\tau p)}$ with $\bar{\alpha}_t \in \mathbb{R}^p$. Here, $\bar{\mathbf{I}}_{p \times q} \in \mathbb{R}^{p \times q}$ is a matrix with all entries as 1, and I_n is the identity matrix of dimension n . By (13) and (17), $\mathbf{W} \sim \mathcal{N}(\bar{\mathbf{I}}_{\tau \times 1} \otimes \bar{\mu}_{\mathbf{w}}, I_\tau \otimes \Sigma_{\mathbf{w}})$ [31, Corollary 1.22]. From (12), we see that for $\bar{\beta} \in \mathbb{R}^n$,

$$\begin{aligned} \Psi_{\mathbf{x}}(\bar{\beta}; \tau, \bar{x}_0) &= \exp\left(j\bar{\beta}^\top (A^\tau \bar{x}_0)\right) \Psi_{\mathbf{W}}(\mathcal{C}_{n \times (\tau p)}^\top \bar{\beta}) \\ &= \exp\left(j\bar{\beta}^\top (A^\tau \bar{x}_0 + \mathcal{C}_{n \times (\tau p)}(\bar{\mathbf{I}}_{\tau \times 1} \otimes \bar{\mu}_{\mathbf{w}}))\right) \times \\ &\quad \exp\left(\frac{\bar{\beta}^\top \mathcal{C}_{n \times (\tau p)}(I_\tau \otimes \Sigma_{\mathbf{w}})\mathcal{C}_{n \times (\tau p)}^\top \bar{\beta}}{2}\right). \end{aligned} \quad (18)$$

Equation (18) is the characteristic function of a multivariate Gaussian random vector [31, Corollary 1.22], and we obtain $\mu_G[\tau]$ and $\Sigma_G[\tau]$ using (17). ■

Proposition 1 shows how Theorem 1 can be used to provide explicit expressions for FSRPD when the characteristic function $\Psi_{\mathbf{x}}(\bar{\beta}; \tau, \bar{x}_0)$ has the structure of known Fourier transforms. In systems where this is not true, the evaluation of (13) can be done via numerical integration. In contrast to the iterative method proposed in Subsection 3.1, Theorem 1 requires a single numerical integration for every time instant of interest $t \in [1, T]$.

Our method solves Problem 1 generally, with additional closed-form solutions for the special case of Gaussian noise.

3.3 Convexity results for reachability analysis

For computational tractability, it is useful to study the convexity properties of the FSRPD and the FSR sets. We define the random vector $\mathbf{w}_B = B\mathbf{w}$ with probability density $\psi_{\mathbf{w}_B}$.

Lemma 4. [30, Lemma 2.1] *If $\psi_{\mathbf{w}}$ is a log-concave distribution, then $\psi_{\mathbf{w}_B}$ is a log-concave distribution.*

Theorem 2. *If $\psi_{\mathbf{w}}$ is a log-concave distribution, then the FSRPD $\psi_{\mathbf{x}}(\bar{y}; t, \bar{x}_0)$ of the system (4) is log-concave in \bar{y} for every $t \in [1, T]$.*

Proof: We prove this theorem via induction. First, we need to show that the base case is true, i.e., we need to show that $\psi_{\mathbf{x}}(\bar{y}; 1, \bar{x}_0)$ is log-concave in \bar{y} . From (4) and Lemma 4, we have $\psi_{\mathbf{x}}(\bar{y}; 1, \bar{x}_0) = \psi_{\mathbf{w}_B}(\bar{y} - A\bar{x}_0)$ which is log-concave since affine transformations preserve log-concavity.

Assume for induction, $\psi_{\mathbf{x}}(\bar{y}; t, \bar{x}_0)$ is log-concave in \bar{y} for some $t \in [1, T]$. From [28, Example 8.9], $\psi_{A\mathbf{x}}(\bar{y}; t, \bar{x}_0) = |A|^{-1} \psi_{\mathbf{x}}(A^{-1}\bar{y}; t, \bar{x}_0)$. Therefore, $\psi_{A\mathbf{x}}(\bar{y}; t, \bar{x}_0)$ is log-concave in \bar{y} since affine transformations preserve log-concavity. Since log-concavity is also preserved under convolution [34, Section 3.5.2], Lemma 4 and (10) completes the proof. ■

Theorem 2 shows that the forward stochastic reachability analysis proposed in Subsections 3.1 and 3.2 yield log-concave FSRPD. Further, we recall that log-concavity implies quasi-concavity [34, Section 3.5]. Quasi-concave functions are functions whose super-level sets are convex [34, Section 3.4]. We have the following corollary from Theorem 2.

Corollary 1. *Superlevel sets of $\psi_{\mathbf{x}}(\cdot; t, \bar{x}_0)$ are convex.*

Theorem 3. *If $\psi_{\mathbf{w}}$ is a log-concave distribution, $\text{FSReach}(t, \bar{x}_0)$ of the system (4) is convex for every $t \in [1, T]$, $\bar{x}_0 \in \mathcal{X}$.*

Proof: Follows from Lemma 3 and Corollary 1. ■

Theorem 3 provides a sufficient condition for the convexity of the FSR set. Theorems 2 and 3 solves Problem 2.a.

We note that integration of log-concave sets over convex sets preserve log-concavity [34, Section 3.5.2]. Therefore, optimization problems involving maximizing the probability over convex sets subject to convex constraints are convex optimization problems via Theorem 2 and Corollary 1.

4. REACHING A NON-ADVERSARIAL TARGET WITH STOCHASTIC DYNAMICS

In this section, we will leverage the theory developed in this paper to solve Problem 2 efficiently.

We consider the problem of a controlled robot (R) having to capture a stochastically moving non-adversarial target, denoted here by a goal robot (G). The robot R has controllable linear dynamics while the robot G has uncontrollable linear dynamics, perturbed by an absolutely continuous random vector. The robot R is said to capture robot G if the robot G is inside a pre-determined set defined around the current position of robot R. We seek an *open-loop* controller (independent of the current state of robot G) for the robot R which maximizes the probability of capturing robot G within the time horizon T . The information available to solve this problem are the position of the robots R and G at $t = 0$, the deterministic dynamics of the robot R, and the stochastic dynamics of the robot G. We consider a 2-D environment, but our approach can be easily extended to higher dimensions. We perform the forward stochastic reachability analysis in the inertial coordinate frame.

We model the robot R as a point mass system discretized in time,

$$\bar{x}_R[t+1] = \bar{x}_R[t] + B_R \bar{u}_R[t] \quad (19)$$

with state (position) $\bar{x}_R[t] \in \mathbb{R}^2$, input $\bar{u}_R[t] \in \mathcal{U} \subseteq \mathbb{R}^2$, input matrix $B_R = T_s I_2$ and sampling time T_s . We define an open-loop control policy $\bar{u}_R[t] = \pi_{\text{open}}(t; \bar{x}_R[0])$ where $\pi_{\text{open}}(\cdot; \bar{x}_R[0])$ depends on the initial condition, that is, $\pi_{\text{open}} : [0, T-1] \times \mathcal{X} \rightarrow \mathcal{U}$ is a sequence of control actions for a given initial condition $\bar{x}_R[0]$. Let \mathcal{M} denote the set of all feasible control policies π_{open} . From (5),

$$\bar{x}_R[\tau+1] = \bar{x}_R[0] + (\bar{1}_{1 \times t} \otimes B_R) \bar{\pi}_\tau, \quad \tau \in [0, T-1] \quad (20)$$

where the input vector is $\bar{\pi}_\tau = [\bar{u}_R^T[0] \ \bar{u}_R^T[1] \ \dots \ \bar{u}_R^T[\tau]]^T \in \bar{\mathcal{M}}_\tau \subseteq \mathbb{R}^{(m\tau)}$ with $m = 2$ and $\bar{u}_R[t] = \pi_{\text{open}}(t; \bar{x}_R[0])$.

We consider two cases for the dynamics of the robot G: 1) point mass dynamics, and 2) double integrator dynamics, both discretized in time and perturbed by an absolutely continuous random vector. In the former case, we presume that the velocity is drawn from a bivariate Gaussian distribution,

$$\mathbf{x}_G[t+1] = \mathbf{x}_G[t] + B_{G,\text{PM}} \mathbf{v}_G[t] \quad (21a)$$

$$\mathbf{v}_G[t] \sim \mathcal{N}(\bar{\mu}_G^{\mathbf{v}}, \Sigma_G). \quad (21b)$$

The state (position) is the random vector $\mathbf{x}_G[t]$ in the probability space $(\mathcal{X}, \sigma(\mathcal{X}), \mathbb{P}_{\mathbf{x}_G^{t, \bar{x}_G[0]}})$ with $\mathcal{X} = \mathbb{R}^2$, disturbance matrix $B_{G,\text{PM}} = B_R$, and $\bar{x}_G[0]$ as the known initial state of the robot G. The stochastic velocity $\mathbf{v}_G[t] \in \mathbb{R}^2$ has mean vector $\bar{\mu}_G^{\mathbf{v}}$, covariance matrix Σ_G and the characteristic function with $\bar{\alpha} \in \mathbb{R}^2$ given in (17). In the latter case, we presume that the acceleration in each dimension is an independent exponential random variable such that

$$\mathbf{x}_G[t+1] = A_{G,\text{DI}} \mathbf{x}_G[t] + B_{G,\text{DI}} \mathbf{a}[t] \quad (22a)$$

$$(\mathbf{a}[t])_x \sim \text{Exp}(\lambda_{\text{ax}}), \quad (\mathbf{a}[t])_y \sim \text{Exp}(\lambda_{\text{ay}}) \quad (22b)$$

$$A_{G,\text{DI}} = I_2 \otimes \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, \quad B_{G,\text{DI}} = I_2 \otimes \begin{bmatrix} \frac{T_s^2}{2} \\ \frac{T_s}{1} \end{bmatrix}.$$

The state (position and velocity) is the random vector $\mathbf{x}_G[t]$ in the probability space $(\mathcal{X}_{\text{DI}}, \sigma(\mathcal{X}_{\text{DI}}), \mathbb{P}_{\mathbf{x}_G^{t, \bar{x}_G[0]}})$ with $\mathcal{X}_{\text{DI}} = \mathbb{R}^4$ and $\bar{x}_G[0]$ as the known initial state of the robot G. The stochastic acceleration $\mathbf{a}[t] = [(\mathbf{a}[t])_x \ (\mathbf{a}[t])_y]^T \in \mathbb{R}_+^2 = [0, \infty) \times [0, \infty)$ has the following probability density and characteristic functions ($\bar{z} = [z_1 \ z_2]^T \in \mathbb{R}_+^2, \bar{\alpha} = [\alpha_1 \ \alpha_2]^T \in \mathbb{R}^2$),

$$\psi_{\mathbf{a}}(\bar{z}) = \lambda_{\text{ax}} \lambda_{\text{ay}} \exp(-\lambda_{\text{ax}} z_1 - \lambda_{\text{ay}} z_2) \quad (23)$$

$$\Psi_{\mathbf{a}}(\bar{\alpha}) = \frac{\lambda_{\text{ax}} \lambda_{\text{ay}}}{(\lambda_{\text{ax}} - j\alpha_1)(\lambda_{\text{ay}} - j\alpha_2)}. \quad (24)$$

The characteristic function $\Psi_{\mathbf{a}}(\bar{\alpha})$ is defined using Property P3 and the characteristic function of the exponential given in [27, Section 26].

Formally, the robot R captures robot G if $\mathbf{x}_G[\tau] \in \text{CaptureSet}(\bar{x}_R[\tau])$. In other words, the capture region of the robot R is the $\text{CaptureSet}(\bar{y}) \subseteq \mathcal{X}$ when robot R has the position $\bar{y} \in \mathcal{X}$. The optimization problem to solve Problem 2 is

$$\begin{aligned} \text{ProbA :} \quad & \text{maximize} && \text{CapturePr}_{\bar{\pi}}(\tau, \bar{\pi}_\tau; \bar{x}_R[0], \bar{x}_G[0]) \\ & \text{subject to} && (\tau, \bar{\pi}_\tau) \in [1, T] \times \bar{\mathcal{M}}_\tau \end{aligned}$$

where the decision variables are the time of capture τ and the control policy $\bar{\pi}$, and the objective function $\text{CapturePr}_{\bar{\pi}}(\cdot)$ gives the probability of robot R capturing robot G. By (20), an initial state $\bar{x}_R[0]$ and the control policy $\bar{\pi}_t$ determines a unique $\bar{x}_R[t]$ for every t . Using this observation, we define the objective function $\text{CapturePr}_{\bar{\pi}}(\cdot)$ in (25). We obtain $\psi_{\mathbf{x}_G}$ in (25) using our solution to Problem 1, Theorem 1.

The following problem is equivalent to Problem ProbA [34, Section 4.1.3],

$$\begin{aligned} \text{ProbB :} \quad & \text{maximize} && \text{CapturePr}_{\bar{x}_R}(\tau, \bar{x}_R[\tau]; \bar{x}_G[0]) \\ & \text{subject to} && \begin{cases} \tau \in [1, T] \\ \bar{x}_R[\tau] \in \text{Reach}_R(\tau; \bar{x}_R[0]) \end{cases} \end{aligned}$$

where the decision variables are the time of capture τ and the position of the robot R $\bar{x}_R[\tau]$ at time τ . From (20), we define the reach set for the robot R at time τ as

$$\text{Reach}_R(\tau; \bar{x}_R[0]) = \{\bar{y} \in \mathcal{X} \mid \exists \bar{\pi}_\tau \in \bar{\mathcal{M}}_\tau \text{ s.t. } \bar{x}_R[\tau] = \bar{y}\}.$$

Several deterministic reachability computation tools are available for the computation of $\text{Reach}_R(\tau; \bar{x}_R[0])$, like MPT [35] and ET [36]. We will now formulate Problem ProbB as a

$$\begin{aligned} \text{CapturePr}_{\bar{x}}(\tau, \bar{\pi}_\tau; \bar{x}_R[0], \bar{x}_G[0]) &= \text{CapturePr}_{\bar{x}_R}(\tau, \bar{x}_R[\tau]; \bar{x}_G[0]) = \mathbb{P}_{\mathbf{x}_G}^{\tau, \bar{x}_G[0]} \{ \mathbf{x}_G[\tau] \in \text{CaptureSet}(\bar{x}_R[\tau]) \} \\ &= \int_{\text{CaptureSet}(\bar{x}_R[\tau])} \psi_{\mathbf{x}_G}(\bar{y}; \tau) d\bar{y}. \end{aligned} \quad (25)$$

convex optimization problem based on the results developed in Subsection 3.3.

Lemma 5. [12] *If the input space \mathcal{U} is convex, the forward reach set $\text{Reach}_R(\tau; \bar{x}_R[0])$ is convex.*

Proposition 2. *If $\psi_{\mathbf{w}}$ is a log-concave distributions and $\text{CaptureSet}(\bar{y})$ is convex for all $\bar{y} \in \mathcal{X}$, then $\text{CapturePr}_{\bar{x}_R}(t, \bar{y}; \bar{x}_G[0])$ is log-concave in \bar{y} for all t .*

Proof: From Theorem 2, we know that $\psi_{\mathbf{x}}(\bar{y}; t, \bar{x}_R[0])$ is log-concave in \bar{y} for every t . The proof follows from (25) since the integration of a log-concave function over a convex set is log-concave [34, Section 3.5.2]. ■

Remark 1. *The density $\psi_{\mathbf{v}}$ is log-concave since multivariate Gaussian density is log-concave [30, Section 2.3]. The density $\psi_{\mathbf{a}}$ is log-concave since exponential distribution (gamma distribution with shape parameter $p = 1$) is log-concave [30, Section 1.4], and log-concavity is preserved for products [34, Section 3.5.2].*

For any $\tau \in [1, T]$, Proposition 2 and Lemma 5 ensures that

$$\begin{aligned} \text{ProbC :} \quad & \text{minimize} && -\log(\text{CapturePr}_{\bar{x}_R}(\tau, \bar{x}_R[\tau]; \bar{x}_G[0])) \\ & \text{subject to} && \bar{x}_R[\tau] \in \text{Reach}_R(\tau; \bar{x}_R[0]) \end{aligned}$$

is a convex optimization problem with the decision variable as $\bar{x}_R[\tau]$. Problem ProbC is an equivalent convex optimization problem of the partial maximization with respect to $\bar{x}_R[t]$ of Problem ProbB since we have transformed the original objective function with a monotone function to yield a convex objective and the constraint sets are identical [34, Section 4.1.3]. Further, any local minimum of Problem ProbC is a global optimum [34, Section 4.2.2]. We solve Problem ProbB by solving Problem ProbC for each time instant $\tau \in [1, T]$ and computing the maximum of the resulting finite set. Note that for numerical stability, Corollary 1 and Theorem 3 ensures that adding an additional constraint to Problem ProbC

$$\text{CapturePr}_{\bar{x}_R}(\tau, \cdot; \bar{x}_G[0]) \geq \epsilon \quad (26)$$

does not affect its convexity (ϵ is a small positive number).

Obtaining the open-loop controller to drive the robot R from $\bar{x}_R[0]$ to $\bar{x}_R^*[\tau^*]$ can be posed as Problem ProbD using the optimal solution of Problem ProbB, $(\tau^*, \bar{x}_R[\tau^*])$. Defining $\mathcal{C}_R = (\bar{I}_{1 \times (\tau^*-1)} \otimes B_R)$ from (20),

$$\begin{aligned} \text{ProbD :} \quad & \text{minimize} && J_\pi(\bar{\pi}_{\tau^*}) \\ & \text{subject to} && \begin{cases} \bar{\pi}_{\tau^*} \in \overline{\mathcal{M}}_{\tau^*} \\ \mathcal{C}_R \bar{\pi}_{\tau^*} = \bar{x}_R[\tau^*] - \bar{x}_R[0] \end{cases} \end{aligned}$$

where the decision variable is $\bar{\pi}_{\tau^*}$. The objective function $J_\pi(\bar{\pi}) = 0$ provides a feasible open-loop controller, and $J_\pi(\bar{\pi}) = \bar{\pi}^\top \bar{R} \bar{\pi}$, $\bar{R} \in \mathbb{R}^{(2\tau^*) \times (2\tau^*)}$ provides an open-loop controller policy that minimizes the control effort while ensuring that maximum probability of robot R capturing robot G is achieved. Solving the optimization problems ProbB and ProbD answers Problem 2.

Our approach to solving Problem 2 is based on our solution to Problem 1, the Fourier transform based forward stochastic reachability analysis, and Problem 2.a, the convexity results of the FSRPD and the FSR sets presented in this paper. In contrast, the iterative approach for the reachability analysis presented in Subsection 3.1 would yield erroneous $\text{CapturePr}_{\bar{x}_R}(\tau, \cdot; \bar{x}_G[0])$ for larger values of τ due to the reliance on numerical integration techniques [32]. Additionally, the traditional approach of dynamic programming based computations [6] would be prohibitively costly for the large FSR sets encountered in this problem due to unbounded disturbances.

All computations in this paper were performed using MATLAB on an Intel Core i7 CPU with 2.10GHz clock rate and 8 GB RAM. The MATLAB code to generate the figures in this section will be uploaded soon at <http://www.unm.edu/~oishi/data/HSCC2017.zip>.

4.1 Goal robot with point mass dynamics

We solve Problem ProbB for the system given by (21). Here, the disturbance set is $\mathcal{W} = \mathbb{R}^2$.

Lemma 6. *For the system given in (21) and initial state of the robot G as $\bar{x}_G[0] \in \mathbb{R}^2$, $\text{FSR}_{\text{Reach}_G}(t, \bar{x}_G[0]) = \mathbb{R}^2$ for every t .*

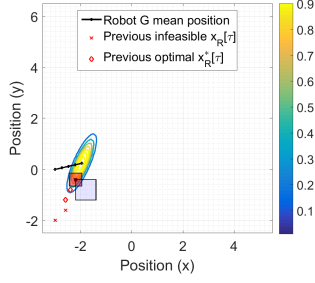
Proof: Follows from Proposition 1 and Lemma 3. ■

Proposition 1 provides the FSRPD and Lemma 6 provides the FSR set for the system (21). The probability of successful capture of the robot G can be computed using (25) since the FSRPD $\psi_{\mathbf{x}_G}(\cdot; t, \bar{x}_G[0])$ is available.

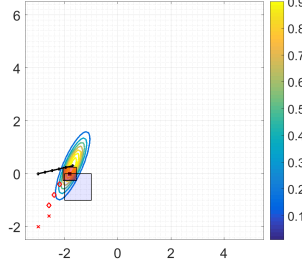
We implement the problem with the following parameters: $T_s = 0.2$, $T = 20$, $\bar{\mu}_G^v = [1.3 \ 0.3]^\top$, $\Sigma_G = \begin{bmatrix} 0.5 & 0.8 \\ 0.8 & 2 \end{bmatrix}$, $\bar{x}_G[0] = [-3 \ 0]^\top$, $\bar{x}_R[0] = [-3 \ -2]^\top$ and $\mathcal{U} = [1, 2]^2$. The capture region of the robot R is a box centered about the position of the robot \bar{y} with side $a = 0.5$ and edges parallel to the axes — $\text{CaptureSet}(\bar{y}) = \text{Box}(\bar{y}, a)$, a convex set. Problem ProbC was solved using *fmincon*, and the objective function $\text{CapturePr}_{\bar{x}_R}(\cdot)$ was implemented using *mvncdf*. We solve Problem ProbD using CVX [37]. The overall computation of Problem ProbB and ProbD took 18.14 seconds with no offline computations. Since Proposition 1 provides explicit expressions for the FSRPD, the evaluation of the FSRPD for any given point $\bar{y} \in \mathcal{X}$ took 5 microseconds on average.

Figure 1 shows the evolution of the mean position of the robot G and the optimal capture position for the robot R at time instants 4, 5, 8, 14, and 20. The contour plots of $\psi_{\mathbf{x}_G}(\cdot; t)$ are rotated ellipses since Σ_E is not a diagonal matrix. From (15), the mean position of the robot G moves in a straight line $\mu_G[t]$, as it is the trajectory of (21a) when the input is always $\bar{\mu}_G^v$. The optimal time of capture is $\tau^* = 5$, the optimal capture position is $\bar{x}_R^*[\tau^*] = [-2.2 \ -0.4]^\top$, and the corresponding probability of robot R capturing robot G is 0.219. Note that at this instant, the reach set of the robot R does not cover the current mean position of the robot G,

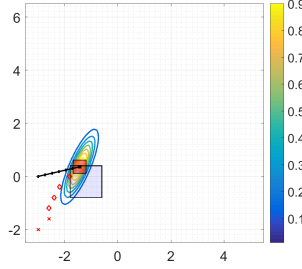
Time = 4
 $\text{CapturePr}_{\bar{x}_R}^* = 0.1571$



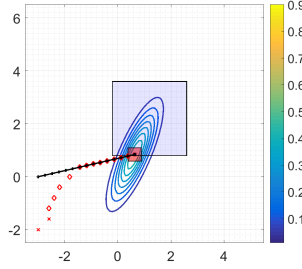
Time = 5
 $\text{CapturePr}_{\bar{x}_R}^* = 0.219$



Time = 6
 $\text{CapturePr}_{\bar{x}_R}^* = 0.2124$



Time = 14
 $\text{CapturePr}_{\bar{x}_R}^* = 0.1049$



Time = 20
 $\text{CapturePr}_{\bar{x}_R}^* = 0.0624$

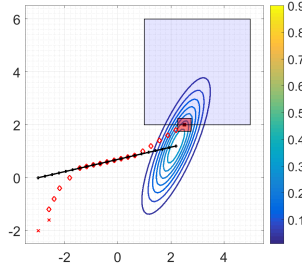


Figure 1: Snapshots of optimal capture positions of the robots G and R when G has point mass dynamics (21). The blue line shows the mean position trajectory of robot G $\mu_G[t]$, the contour plot characterizes $\psi_{x_G}(\cdot; t, \bar{x}_G[0])$, the blue box shows the reach set of the robot R at time t $\text{Reach}_R(t, \bar{x}_R[0])$, and the red box shows the capture region centered at $\bar{x}_R[\tau^*]$ $\text{CaptureSet}(\bar{x}_R[\tau^*])$.

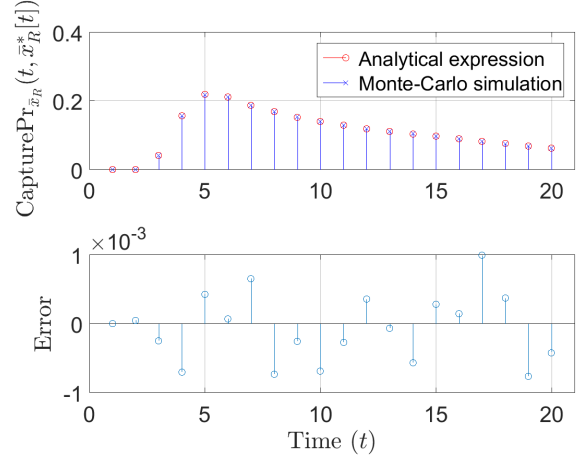


Figure 2: Solution to Problem ProbC for robot G dynamics in (21), and validation of $\text{CapturePr}_{\bar{x}_R}(\tau, \bar{x}_R^*[\tau]; \bar{x}_G[0])$ via Monte-Carlo simulations. The optimal capture time is $\tau^* = 5$ and the likelihood of capture is $\text{CapturePr}_{\bar{x}_R}(\tau^*, \bar{x}_R^*[\tau^*]; \bar{x}_G[0]) = 0.219$.

$\bar{\mu}[\tau^*] = [-1.96 \ -0.24]^\top$ (Figure 1b). While the reach set covers the mean position of robot G at the next time instant $t = 6$, the uncertainty in (21) causes the probability of successful capture to reduce (Figure 1c). Hence, the intuitive answer of attempting to reach the $\mu_G[t]$ need not always be correct.

Figure 2 shows the optimal capture probabilities obtained when solving Problem ProbC for the dynamics (21). We further validate the result for $\text{CapturePr}_{\bar{x}_R}(\tau, \bar{x}_R^*[\tau]; \bar{x}_G[0])$ using Monte-Carlo simulation of robot G with 500,000 particles.

4.2 Goal robot with double integrator dynamics

We now consider a more complicated capture problem, in which the disturbance is exponential (hence tracking the mean has little relevance because it does not have the highest likelihood), and the robot dynamics are more realistic. We solve Problem ProbB for the system given by (22). Here, the disturbance set is $\mathcal{W} = \mathbb{R}_+^2$. Based on the mean of the stochastic acceleration $\mathbf{a}[t]$, the robot G has a parabolic trajectory due to the double integrator dynamics, as opposed to the linear trajectory seen in Subsection 4.1. Also, in this case, we do not have an explicit expression for the FSRPD like Proposition 1. Using Theorem 1, we obtain an explicit expression for the characteristic function of the FSRPD. We utilize Lemma 1 to evaluate the objective function of Problem ProbB.

Analogous to Lemma 6 and Proposition 1, we characterize the FSR set in Lemma 7 and the FSRPD in Proposition 3.

Lemma 7. *For the system given in (22) with initial state $\bar{x}_G[0] \in \mathbb{R}^4$ of the robot G, we have $\text{FSR}_{\text{Reach}_G}(t, \bar{x}_G[0]) = \{A_{G, \text{DI}}^t \bar{x}_G[0]\} \oplus \mathbb{R}_+^4$ for every $t \geq 2$, and $\text{FSR}_{\text{Reach}_G}(1, \bar{x}_G[0]) = \{A_{G, \text{DI}} \bar{x}_G[0]\} \oplus B_{G, \text{DI}} \mathbb{R}_+^2$.*

Proof: For the dynamics in (22), $\mathcal{C}_{4 \times (2t)}^\top \mathbb{R}_+^2 = \mathbb{R}_+^4$ since the rank of $\mathcal{C}_{4 \times (2t)}^\top$ is 4 for every $t \geq 2$, and elements of $\mathcal{C}_{4 \times (2t)}^\top$ are nonnegative. For $t = 1$, $\mathcal{C}_{4 \times (2t)}^\top = B_{G, \text{DI}}$.

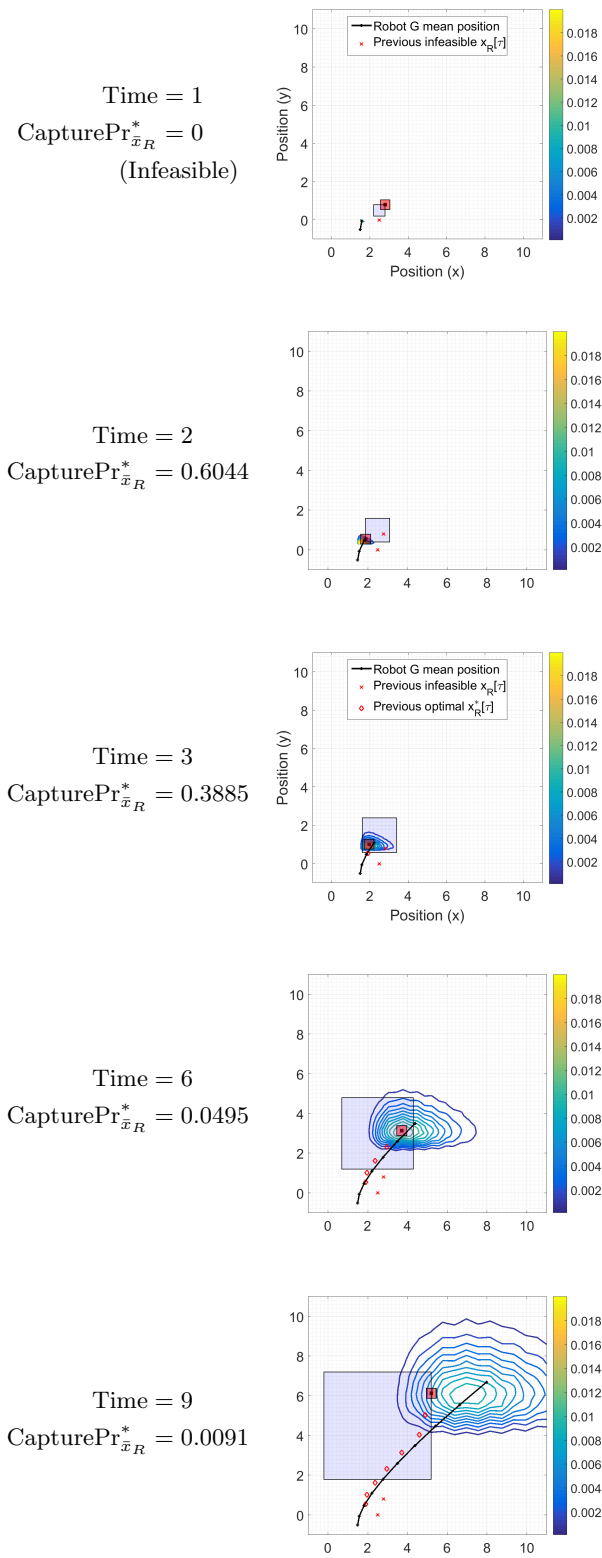


Figure 3: Snapshots of optimal capture positions of the robots G and R when G has double integrator dynamics (22). The blue line shows the mean position trajectory of robot G $\mu_G[t]$, the contour plot characterizes $\psi_{\mathbf{x}_G}^{\text{pos}}(\cdot; t, \bar{x}_G[0])$ via Monte-Carlo simulation of 500,000 particles, the blue box shows the reach set of the robot R at time t , $\text{Reach}_R(t, \bar{x}_R[0])$, and the red box shows the capture region centered at $\bar{x}_R[\tau^*]$, $\text{CaptureSet}(\bar{x}_R[\tau^*])$.

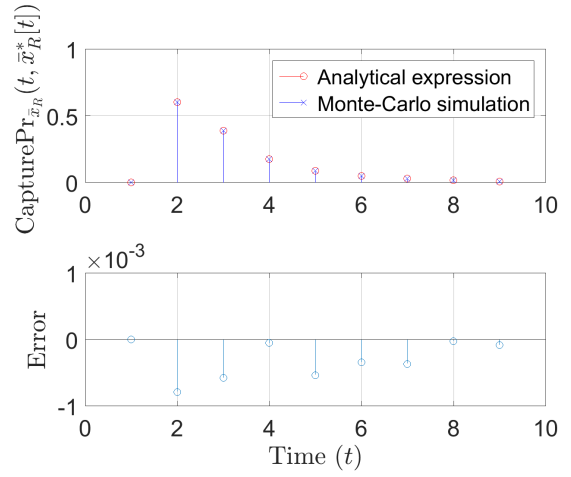


Figure 4: Solution to Problem ProbC for robot G dynamics in (22), and validation of $\text{CapturePr}_{\bar{x}_R}(\tau, \bar{x}_R^*[\tau]; \bar{x}_G[0])$ via Monte-Carlo simulations. The optimal capture time is $\tau^* = 2$ and the capture probability is $\text{CapturePr}_{\bar{x}_R}(\tau^*, \bar{x}_R^*[\tau^*]; \bar{x}_G[0]) = 0.6044$.

Lemma 2 completes the proof. \blacksquare

Proposition 3. *The characteristic function of the FSRPD of the robot G for dynamics (22) is*

$$\Psi_{\mathbf{x}_G}(\bar{\beta}; \tau, \bar{x}_G[0]) = \exp(j\bar{\beta}^\top (A_{G, \text{DI}}^\tau \bar{x}_G[0])) \times \prod_{t=0}^{\tau-1} \frac{\lambda_{\text{ax}} \lambda_{\text{ay}}}{(\lambda_{\text{ax}} - j\bar{\alpha}_{2t})(\lambda_{\text{ay}} - j\bar{\alpha}_{2t+1})} \quad (27)$$

where $\bar{\alpha} = \mathcal{C}_{4 \times (2\tau)}^\top \bar{\beta} \in \mathbb{R}^{(2\tau)}$ and $\bar{\beta} \in \mathbb{R}^4$. The FSRPD of the robot G is $\psi_{\mathbf{x}_G}(\bar{x}; t, \bar{x}_G[0]) = \mathcal{F}^{-1}\{\Psi_{\mathbf{x}_G}(\cdot; t, \bar{x}_G[0])\}(-\bar{x})$.

Proof: Application of Theorem 1 to the dynamics (22). \blacksquare

Similarly to the case in Subsection 4.1, Proposition 3 provides the FSRPD and Lemma 7 provides the FSR set of the system (22).

To solve Problem ProbB, we define $\text{CapturePr}_{\bar{x}_R}(\cdot)$ as in (31). Since we are interested only in the position of robot G, we require only the marginal density of the FSRPD over the position subspace of robot G, $\psi_{\mathbf{x}_G}^{\text{pos}}$. By Property P4, we have for $\bar{\gamma} = [\gamma_1 \ \gamma_2] \in \mathbb{R}^2$,

$$\Psi_{\mathbf{x}_G}^{\text{pos}}(\bar{\gamma}; t, \bar{x}_G[0]) = \Psi_{\mathbf{x}_G}([\gamma_1 \ 0 \ \gamma_2 \ 0]^\top; t, \bar{x}_G[0]). \quad (28)$$

Unlike the case with Gaussian disturbance, explicit expressions for the FSRPD $\psi_{\mathbf{x}_G}$ or its marginal density $\psi_{\mathbf{x}_G}^{\text{pos}}$ are not available since the Fourier transform (27) is not standard.

Lemma 8. *The FSRPD $\psi_{\mathbf{x}_G}(\bar{x}; t, \bar{x}_G[0]) \in L^2(\mathbb{R}^4)$.*

Proof: Use (10) to prove this lemma via induction (similar to the proof of Theorem 2). Note that functions in $L^1(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$ are closed under convolution [31, Theorem 1.3]. \blacksquare

Lemma 9. *The marginal density $\psi_{\mathbf{x}_G}^{\text{pos}}(\bar{x}; t, \bar{x}_G[0]) \in L^2(\mathbb{R}^2)$.*

Proof: From Lemma 8 and Property P5, we know that $\Psi_{\mathbf{x}_G} \in L^2(\mathbb{R}^4)$. Since $\Psi_{\mathbf{x}_G}^{\text{pos}} \in L^2(\mathbb{R}^2)$ from (28), $\psi_{\mathbf{x}_G}^{\text{pos}} \in L^2(\mathbb{R}^2)$ via Property P5. \blacksquare

We define a convex set $\text{CaptureSet}(\bar{y}_R, a) = \text{Box}(\bar{y}_R, a) \subseteq \mathbb{R}^2$ as the capture region, where $\bar{y}_R \in \mathbb{R}^2$ is the state of the robot R. We define a function $h(\bar{y}; \bar{y}_R, a) = \mathbf{1}_{\text{Box}(\bar{y}_R, a)}(\bar{y})$. Note that $h(\cdot; \bar{y}_R, a)$ is a 2-D box function centered at \bar{y}_R with edge length $a > 0$. The Fourier transform of h is (using Property P2 and [38, Chapter 13])

$$\begin{aligned} H(\bar{\gamma}; \bar{y}_R, a) &= \mathcal{F}\{h(\cdot; \bar{y}_R, a)\}(\bar{\gamma}) \\ &= 4a^2 \exp(-j\bar{\gamma}_R^\top \bar{\gamma}) \frac{\sin(a\gamma_1) \sin(a\gamma_2)}{\gamma_1 \gamma_2}. \end{aligned} \quad (29)$$

Clearly, h is square-integrable, and from Lemmas 1 and 9, we define $\text{CapturePr}_{\bar{x}_R}(\cdot)$ in (31). Equation (31) is evaluated using (27), (28), and (29). We use (31) as opposed (30) due to the unavailability of an explicit expression for $\psi_{\bar{x}_G}^{\text{pos}}$. The numerical evaluation of the inverse Fourier transform of $\Psi_{\bar{x}_G}^{\text{pos}}$ to compute (30) will result in more approximation error as compared to (31) because of the two numerical integrations involved. Hence, Lemmas 1 and 9 enable an accurate evaluation of $\text{CapturePr}_{\bar{x}_R}(\cdot)$.

We implement the following parameters: $T_s = 0.2$, $T = 9$, $\lambda_{\text{ax}} = 0.25$, $\lambda_{\text{ay}} = 0.45$, $\bar{x}_G[0] = [1.5 \ 0 \ -0.5 \ 2]^\top$, $\bar{x}_R[0] = [2.5 \ 0]^\top$ and $\mathcal{U} = [-1.5, 1.5] \times [1, 4]$. The capture region of the robot R is a box centered about the position of the robot \bar{y} with side $a = 0.5$ and edges parallel to the axes: $\text{CaptureSet}(\bar{y}) = \text{Box}(\bar{y}, a)$. Problem ProbC was solved using *fmincon*, and the objective function $\text{CapturePr}_{\bar{x}_R}(\cdot)$ was implemented using the numerical integration of (31). We solve Problem ProbD using CVX [37]. The overall computation of Problem ProbB and ProbD took 615.27 seconds (~ 10 minutes) with no offline computations. The computational time increased because the improper integral (31) was evaluated numerically. The evaluation of the FSRPD for any given point $\bar{y} \in \mathcal{X}$ takes about 50 seconds due to the numerical evaluation of the improper integral (2), and the runtime and the accuracy depends heavily on the point \bar{y} as well as the bounds used for the integral. The evaluation of $\text{CapturePr}_{\bar{x}_R}(\cdot)$ is faster because $H(\bar{\gamma}; \bar{y}_R, a)$ is a decaying, 2-D sinc function.

Figure 3 shows the evolution of the mean position of the robot G and the optimal capture position for the robot R at time instants 1, 2, 3, 6, and 9. The contour plots of $\psi_{\bar{x}_G}^{\text{pos}}(\cdot; t, \bar{x}_G[0])$ were computed via Monte-Carlo simulation using 500,000 particles since evaluating $\psi_{\bar{x}_G}^{\text{pos}}(\cdot; t, \bar{x}_G[0])$ via (2) over a grid is computationally expensive. Note that the mean position of the robot G does not coincide with the mode of $\psi_{\bar{x}_G}^{\text{pos}}(\cdot; t, \bar{x}_G[0])$ in contrast to the problem discussed in Subsection 4.1. The optimal time of capture is at $\tau^* = 2$, the optimal capture position is $\bar{x}_R^*[\tau^*] = [1.9 \ 0.55]^\top$, and the corresponding probability of robot R capturing robot G is 0.6044 (Figure 3b).

Figure 4 shows the optimal capture probabilities obtained when solving Problem ProbC for the dynamics (22). We further validate the result for $\text{CapturePr}_{\bar{x}_R}(\tau, \bar{x}_R^*[\tau]; \bar{x}_G[0])$ using Monte-Carlo simulation of robot G with 500,000 particles.

5. CONCLUSIONS AND FUTURE WORK

This paper provides a method for forward stochastic reachability analysis using Fourier transforms. The method is applicable to uncontrolled stochastic linear systems. Fourier transforms simplify the computation and mitigate the curse of dimensionality. We also analyze several convexity results

associated with the FSRPD and FSR sets. We demonstrate our method on the problem of controller synthesis for a controlled robot attempting to capture a stochastically moving non-adversarial target.

Future work includes the extension to a model predictive control framework and the analysis of this method when applied to discrete random vectors (countable disturbance sets). Extensions to multiple pursuers will also be investigated.

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$$\text{CapturePr}_{\bar{x}_R}(\tau, \bar{x}_R[\tau]; \bar{x}_G[0]) = \int_{\mathbb{R}^2} \psi_{\bar{x}_G}^{\text{pos}}(\bar{x}; \tau, \bar{x}_G[0]) h(\bar{x}; \bar{x}_R[\tau], a) d\bar{x} \quad (30)$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{R}^2} \Psi_{\bar{x}_G}^{\text{pos}}(\bar{\gamma}; \tau, \bar{x}_G[0]) H(\bar{\gamma}; \bar{x}_R[\tau], a) d\bar{\gamma}. \quad (31)$$

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